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Cyclotron resonance of a polaron

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Abstract. The theory of the mass shift and the linewidth of the cyclotron resonance of a polaron is given. The lineshape function which was obtained earlier by the author on the basis of the path-integral method with an optimised model is utilised. This formula has the particular advantage of being applicable on a unified footing to electron–phonon interactions of any strength, elastic or inelastic. Applications are made to the Fröhlich polaron system and explicit analytical expressions are obtained for the first time for some limiting cases of interest.

1. Introduction

The quantum theory of the cyclotron resonance of electrons coupled to phonons was formulated by Kawabata (1967). His theory, based on Mori's formalism (1966) of the generalised Langevin equation, enables one to treat the case where the electron–phonon interaction is weak and elastic. Theoretical investigations of the cyclotron resonance of piezo-electric polarons were performed subsequently by Saitoh and Kawabata (1967) based on this method, and later by Miyake (1968) in the Green function method.

When the electron–phonon interaction is inelastic such as in the Fröhlich polaron (Fröhlich 1954) where the optical phonon with finite energy is involved, extensive investigations of the energy states in a magnetic field have been performed (see e.g. for review Larsen 1972, Saitoh 1980b, 1981, Arisawa and Saitoh 1981, Peeters and Devreese 1981a, b, 1982a, b, and references contained in these papers), but the study of the dynamical properties of a polaron is mostly limited to the static magneto-resistance in the weak coupling limit (see e.g. for review Kubo *et al* 1965, Gurevich and Firsov 1961, Dworin 1965), except for the calculation of the cyclotron resonance lineshape by Nakayama (1969) when the cyclotron frequency is near the optical phonon energy. When the magnetic field is so high that the Landau quantisation becomes important, theoretical understanding of the cyclotron resonance for either inelastic scatterings or strong couplings, e.g. how the linewidth is related to the lifetime of the Landau states, is far from complete.

In order to fill this gap, in an earlier paper (Saitoh 1982) theoretical attempts were made to formulate magneto-conductivity for the general class of electron–phonon interactions of both elastic and inelastic scatterings and of weak and strong interactions. There, the Feynman path-integral method (Feynman 1955) was employed in which

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all the dynamical variables of the phonons were eliminated at the expense of the complicated non-local (in time) interaction of electrons with themselves, and the resulting non-local interaction was approximately simulated by an optimised model (Saitoh 1980a, 1981, Adamowski *et al* 1980) which contained variational functions with an infinite number of variational parameters. The variational functions were chosen so that the free energy was minimised, and they satisfied a set of nonlinear integral equations. The conductivity was calculated from the thermal–double-time correlation function in the representation of the single path-integral, and was expressed by the memory functions. This formula was applicable to all coupling constants and temperatures once these variational functions were determined. Applications were made to calculate the magneto-resistance of the Fröhlich polaron for both weak and strong coupling cases.

The aim of this paper is to investigate the effective mass and the linewidth of the cyclotron resonance based on the previously obtained general magneto-conductivity expression. Explicit analytical expressions will be given for the case of the Fröhlich polaron for the limiting cases of weak and strong couplings. New results are presented for the case of strong magnetic fields. In order to obtain the results for the intermediate range where either the coupling or the magnetic field is neither very weak nor very strong, numerical analyses are necessary. This is not pursued in this work.

Section 2 gives a brief outline of the resonance lineshape function, and evaluation of the mass shift and the linewidth is made in § 3 for low magnetic fields and in § 4 for high fields. When the magnetic field is low, we recover the semiclassical Drude formula as expected. When the magnetic field is high, on the other hand, the Landau quantisation becomes important, and completely new features come in. For example, the cyclotron mass shift becomes smaller than the bare band mass and the linewidth is no longer a simple combination of lifetimes of the two lowest Landau states. The last section is devoted to discussion and summary. In the appendix, an expression of the memory function in the weak coupling limit is given for the general form of the electron–phonon interaction.

2. Formula for the absorption line-shape

When a static magnetic field $\boldsymbol{H}$ is applied along the z axis of the system, and a circularly polarised electric field with angular frequency ω is introduced along $\boldsymbol{H}$, the absorption lineshape of the cyclotron resonance is proportional to the magneto-conductivity

$$\mathrm{Re}\,\sigma_{+-}(\omega)=\mathrm{Re}[\sigma_{xx}(\omega)-\mathrm{i}(\sigma_{xy}(\omega)-\sigma_{yx}(\omega))+\sigma_{yy}(\omega)] \qquad (2.1)$$

where $\sigma_{ij}(\omega)$ is the conductivity tensor with i and j standing for the components of the Cartesian coordinates. In a previous paper (Saitoh 1981), we developed the conductivity formula for an electron coupled to phonons through the interaction Hamiltonian

$$\mathscr{H}_{\mathrm{el}}=\sum_{\boldsymbol{q}}\Gamma_q(b_{\boldsymbol{q}}+b_{\boldsymbol{q}}^{+})\,\mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \qquad (2.2)$$

where Γ_q is the coupling function, $b_{\boldsymbol{q}}$ and $b_{\boldsymbol{q}}^{+}$ are, respectively, destruction and creation operators of a phonon with wavevector $\boldsymbol{q}$ and angular frequency Ω_q, and $\boldsymbol{r}$ is the position vector of an electron. We have evaluated the thermal–double-time correlation function in the path-integral representation by using the optimised model where a trial action is used with an infinite number of variational parameters which are determined by the minimisation principle of free energy. The magneto-conductivity tensor can be expressed as the analytic continuation of the thermal correlation function thus obtained. The final result is given by

$$\mathrm{Re}\,\sigma_{+-}(\omega) = (2N_e e^2\hbar/m)\,\mathrm{Re}[\mathrm{i}(\omega-\omega_c)+M_\perp(\omega)]^{-1} \tag{2.3}$$

where N_e is the electron number density, $-e$ the electronic charge, m the bare band mass of an electron, $\omega_c = |e|H/mc$ the cyclotron frequency and $M_\perp(\omega)$ the memory function. The memory function in general satisfies

$$M_\perp(\omega) = M_\perp^*(-\omega), \qquad \mathrm{Re}\,M_\perp(\omega) \geq 0. \tag{2.4}$$

In our particular approximation, the memory function has the form

$$M_\perp(\omega) = \frac{2\mathrm{i}}{\hbar^2\omega}\left(\int_0^{\beta/2} \mathrm{d}\tau\,(\cosh\hbar\omega\tau - 1)L_\perp(\tau) - \mathrm{i}\sinh\tfrac{1}{2}\hbar\omega\beta\int_0^\infty \mathrm{d}t\,\mathrm{e}^{-\mathrm{i}\hbar\omega t}L_\perp(\mathrm{i}t+\tfrac{1}{2}\beta)\right) \tag{2.5}$$

where β is the inverse temperature, and the function $L_\perp(\tau)$ is defined by

$$L_\perp(\tau) = \sum_q \frac{\hbar^2 q_\perp^2}{2m}|\Gamma_q|^2\mathcal{N}_q(\tau)\exp\left(-\frac{\hbar^2 q_\perp^2}{2m}\Lambda_\perp(\tau) - \frac{\hbar^2 q_z^2}{2m}\Lambda_\parallel(\tau)\right). \tag{2.6}$$

Here, $q_\perp$, q_z are the wavevectors' perpendicular and parallel components to the magnetic field, and

$$\mathcal{N}_q(\tau) = (N_q+1)\,\mathrm{e}^{-\hbar\Omega_q\tau} + N_q\,\mathrm{e}^{\hbar\Omega_q\tau} = \cosh[\hbar\Omega_q(\tau-\beta/2)]/\sinh(\hbar\Omega_q\beta/2) \tag{2.7}$$

with N_q the Planck distribution function for a phonon with wavevector q and angular frequency Ω_q, and $\Lambda_\perp(\tau)$ and $\Lambda_\parallel(\tau)$ are symmetric functions such that $\Lambda_\perp(\tau) = \Lambda_\perp(\beta-\tau)$ etc, and satisfy a set of coupled equations (2.9), (2.10), (2.14) and (2.18) of Saitoh (1982).

For the explicit evaluation of $M_\perp(\omega)$, we need to know these solutions $\Lambda_\perp(\tau)$ and $\Lambda_\parallel(\tau)$ of the coupled equations. In some limiting cases of interest, analytic forms are available. In the weak coupling limit of the electron–phonon interaction, we can replace $\Lambda_\perp(\tau)$ and $\Lambda_\parallel(\tau)$ in (2.6) by their zeroth-order approximants $\Lambda_\perp^{(0)}(\tau)$ and $\Lambda_\parallel^{(0)}(\tau)$ where

$$\Lambda_\perp^{(0)}(\tau) = \frac{\cosh(\hbar\omega_c\beta/2) - \cosh[\hbar\omega_c(\tau-\beta/2)]}{\hbar\omega_c\sinh(\hbar\omega_c\beta/2)}, \tag{2.8}$$

$$\Lambda_\parallel^{(0)}(\tau) = \tau(1-\tau/\beta) \equiv \Lambda^{(0)}(\tau), \tag{2.9}$$

since $L_\perp(\tau)$ itself is already proportional to Γ_q^2.

In the strong coupling case, on the other hand, the functions $\Lambda_\perp(\tau)$ and $\Lambda_\parallel(\tau)$ are dependent on the form of $\Gamma_q^2\mathcal{N}_q(\tau)$, and we do not have a general solution, and have to specify the nature of the interactions to proceed further. The explicit evaluation of the memory function $M_\perp(\omega)$ will be given in the following sections for the case of the Fröhlich polaron.

3. Case of low magnetic fields

When a magnetic field is small, we are allowed to replace $\Lambda_{\perp}(\tau)$ and $\Lambda_{\|}(\tau)$ by their zero magnetic field value $\Lambda(\tau)$, the error being of the order of H^2. The memory function is no longer dependent on the magnetic field in this case, and is given by its zero-field value. Hence, we drop the subscript $\perp$ in this section. Since ω_c is small, we only need to know $M(\omega)$ for small ω. Then the memory function $M(\omega)$ may be expanded as

$$M(\omega) \simeq M(0) + i\omega[\partial \operatorname{Im} M(\omega)/\partial\omega]_{\omega=0} \tag{3.1}$$

and the lineshape takes the form

$$\operatorname{Re} \sigma_{+-}(\omega) = \frac{2N_e e^2 \hbar}{m_t} \frac{\Gamma}{(\omega - eH/m_t c)^2 + \Gamma^2} \tag{3.2}$$

where

$$m/m_t - 1 = [\partial \operatorname{Im} M(\omega)/\partial\omega]_{\omega=0}, \tag{3.3}$$

$$\Gamma = (m/m_t) M(0). \tag{3.4}$$

Here, m_t and Γ^{-1} are the transport mass and the collision time for the DC electric field. Γ should not be confused with Γ_q. This is nothing but the usual semiclassical Drude formula where everything is expressed in terms of the DC values. The formula (3.2) suggests that polarons with effective mass m_t are good physical pictures to describe elementary excitations in low magnetic fields (Larsen 1969, Saitoh 1980b).

In the following, we consider the case of the Fröhlich polaron. In the conventional non-dimensional units such that $\hbar$, the electron band mass m and the optical phonon frequency Ω_q = constant are taken to be unity, the coupling function Γ_q takes the form

$$\Gamma_q^2 = 2\sqrt{2}\pi\alpha / Vq^2 \tag{3.5}$$

where α is the Fröhlich coupling constant and V the volume of the system. In this case, we have from (2.6)

$$L(\tau) = \frac{\alpha}{3\sqrt{\pi}} \frac{\cosh(\tau - \beta/2)}{\sinh(\beta/2)} [\Lambda(\tau)]^{-3/2}. \tag{3.6}$$

When the coupling is weak, i.e. $\alpha \ll 1$, $\Lambda(\tau)$ in (3.6) may be replaced by its zeroth-order approximant (2.9). Putting (3.6) into (2.5), we obtain after some algebra

$$\operatorname{Re} M(\omega) = \frac{2\alpha}{3\omega} \sqrt{\frac{\beta}{\pi}} \frac{\sinh(\beta\omega/2)}{\sinh(\beta/2)} \left[(1+\omega) K_1\left(\frac{\beta}{2}(1+\omega)\right) + |1-\omega| K_1\left(\frac{\beta}{2}|1-\omega|\right) \right], \tag{3.7a}$$

$$\operatorname{Im} M(\omega) = \frac{2\alpha\sqrt{\pi\beta}}{3\omega \sinh(\beta/2)} \left[I_1\left(\frac{\beta}{2}\right) - \frac{1+\omega}{2} e^{-\beta\omega/2} I_1\left(\frac{\beta}{2}(1+\omega)\right) - \frac{1-\omega}{2} e^{\beta\omega S(1-\omega)/2} I_1\left(\frac{\beta}{2}(1-\omega)\right) \right], \tag{3.7b}$$

where I_n and K_n are the modified Bessel functions of the first and second kinds, respectively, and $S(x)$ is the signature function which is equal to either +1 or −1 according as x is positive or negative. These results were implicitly contained in the work of Feynman *et al* (1962) and Devreese (private communication), but have not been given explicitly before. From these results, we obtain

$$m_t - 1 = [\alpha\beta^2\sqrt{\pi\beta}/6\sinh(\beta/2)][I_0(\tfrac{1}{2}\beta)(1-1/\beta) - I_1(\tfrac{1}{2}\beta)], \tag{3.8}$$

$$\Gamma = \tfrac{2}{3}\alpha\beta(\beta/\pi)^{1/2}K_1(\beta/2)/\sinh(\beta/2). \tag{3.9}$$

The asymptotic formulae for very low temperature ($\beta \gg 1$) and for very high temperature ($\beta \ll 1$) are given by

$$m_t - 1 = \begin{cases} \tfrac{1}{6}\alpha(1 + 9/4\beta + \ldots) & (\beta \gg 1), \quad (3.10a) \\ -\tfrac{1}{3}\alpha\sqrt{\pi\beta}(1 - \beta + \ldots) & (\beta \ll 1), \quad (3.10b) \end{cases}$$

$$\Gamma = \begin{cases} \tfrac{4}{3}\alpha\beta\,\mathrm{e}^{-\beta}(1 + 3/4\beta + \ldots) & (\beta \gg 1), \quad (3.11a) \\ 8\alpha/3\sqrt{\pi\beta} & (\beta \ll 1). \quad (3.11b) \end{cases}$$

It is well known that Γ^{-1} at low temperature differs unfortunately from the DC collision time obtained by the Boltzmann equation (Kadanoff 1963, Langreth and Kadanoff 1964, Langreth 1967) by the factor of $3/2\beta$. It should be remarked that the transport mass m_t at low temperature increases with temperature as expected. This temperature dependence is in marked contrast to that of the magnetic mass and the inertial mass (Saitoh 1980a, b, Saitoh and Arisawa 1980, Arisawa and Saitoh 1981) which are decreasing functions of temperature. This is understandable since the latter two masses are defined through the static correlation functions which are related to the real part of the free energy, whereas the transport mass is related to the dissipative part. The thermal averaging procedures of each mass are different and consequently result in the different temperature dependence. At zero temperature, however, all these masses become identical.

Next we consider the strong coupling case. In this case the Feynman model action (Feynman 1955) is a good description (Saitoh 1980a), and the approximate solution to $\Lambda(\tau)$ is given by

$$\Lambda(\tau) = \frac{1}{v^2}\left(\tau\left(1 - \frac{\tau}{\beta}\right) + \frac{v^2 - 1}{v}\,\frac{\cosh(v\beta/2) - \cosh v(\tau - \beta/2)}{\sinh(v\beta/2)}\right), \tag{3.12}$$

where

$$v = 4\alpha^2/9\pi. \tag{3.13}$$

In this case, the calculation proceeds exactly in the same way as the one done by Feynman *et al* (1962). For the first term of $M(\omega)$ of (2.5), we put

$$\Lambda(\tau) = 1/v \tag{3.14}$$

since $v \gg 1$, and for the second term

$$\Lambda(\mathrm{i}t + \beta/2) = (t^2 + a^2)/\beta v^2 \tag{3.15}$$

where

$$a = \{\beta(\tfrac{1}{4}\beta + [(v^2 - 1)/v]\coth\tfrac{1}{2}v\beta)\}^{1/2} \approx [\beta(v + \beta/4)]^{1/2} \tag{3.16}$$

and we have discarded an oscillatory term $\cos vt$ in (3.12) since this contributes to the final result in the negligible order of e^{-av} for $\omega \ll 1$. We have finally

$$M(\omega)=\frac{2\alpha v^{3/2}}{3\sqrt{\pi}}\left(\frac{i\omega}{1-\omega^2}+\frac{\sinh(\beta\omega/2)}{\sinh(\beta/2)}\frac{(v\beta)^{3/2}}{2\omega a}\right.$$
$$\left.\times\{(1+\omega)K_1[a(1+\omega)]+|1-\omega|K_1(a|1-\omega|)\}\right) \qquad (3.17)$$

for $v \gg 1$, $a \gg 1$ and $\omega \ll 1$. Therefore, we recover the well known result for the effective mass (Feynman 1955)

$$m_t = 2\alpha v^{3/2}/3\sqrt{\pi} = 16\alpha^4/81\pi^2. \qquad (3.18)$$

The linewidth differs from Re $M(0)$ by the factor of the effective mass as seen in (3.4) and therefore

$$\Gamma=\frac{\beta(v\beta)^{3/2}}{2a\sinh(\beta/2)}K_1(a)\simeq\frac{\sqrt{\pi}\beta}{\sinh(\beta/2)}\left(\frac{v\beta}{2a}\right)^{3/2}e^{-a} \qquad (3.19)$$

where in the last line the asymptotic formula of $K_1(a)$ for $a \gg 1$ has been used. The linewidth is extremely sharp because of the exponential factor in (3.19). This suggests that the important part of the electron–phonon interaction is taken care of by the renormalisation of the effective mass and the remaining scattering effect is small in the strong coupling limit.

4. High magnetic field case

In this case $\Lambda_\perp(\tau)$ is inversely proportional to ω_c in general and hence much smaller than $\Lambda_\|(\tau)$. By performing the q summation in (2.6), we have for the Fröhlich polaron case

$$L_\perp(\tau)=\frac{\alpha}{4\sqrt{\pi}}\frac{\cosh(\tau-\beta/2)}{\sinh(\beta/2)}\frac{1}{(\Lambda_\|-\Lambda_\perp)^{3/2}}\left(\frac{2\sqrt{\Lambda_\|}\sqrt{\Lambda_\|-\Lambda_\perp}}{\Lambda_\perp}-\ln\frac{\sqrt{\Lambda_\|}+\sqrt{\Lambda_\|-\Lambda_\perp}}{\sqrt{\Lambda_\|}-\sqrt{\Lambda_\|-\Lambda_\perp}}\right)$$
$$\simeq\frac{\alpha}{2\sqrt{\pi}}\frac{\cosh(\tau-\beta/2)}{\sinh(\beta/2)}\frac{1}{\Lambda_\perp(\tau)\sqrt{\Lambda_\|(\tau)}}. \qquad (4.1)$$

In the following, we study the limiting cases of very weak and strong couplings. We first consider the weak coupling limit. Putting (4.1) into (2.5) with the use of (2.8) and (2.9), we have the expression for $M_\perp^{(1)}(\omega)$ for high magnetic fields. Since the resonance occurs at $\omega=\omega_c+O(\alpha)$, we only need to calculate $M_\perp^{(1)}(\omega_c)$. We note the identity

$$\frac{1}{1-y\cosh\tau}=\frac{1}{(1-y^2)^{1/2}}\left[1+2\sum_{n=1}^{\infty}\left(\frac{y}{1+(1-y^2)^{1/2}}\right)^n\cosh n\tau\right] \qquad (4.2)$$

and expand $\cosh[\omega_c(\tau-\beta/2)]$ in $\Lambda_\perp^{(0)}(\tau)$ in the series in $\cosh[n\omega_c(\tau-\beta/2)]$. Then, by changing the variable $\tau=(\beta/2)(1-u)$, and introducing the abbreviation

$$z=\beta\omega_c/2 \qquad (4.3)$$

we have for the first integral in the bracket of (2.5)

$$\int = \omega_c\sqrt{\beta}\int_0^1 du\,\frac{\cosh(\beta u/2)}{(1-u^2)^{1/2}}\left(e^{-z}\cosh z - 1 + 2\sinh^2 z\sum_{m=1}^{\infty} e^{-nz(u+1)}\right)$$

$$= \tfrac{1}{2}\pi\omega_c\sqrt{\beta}\left\{(e^{-z}\cosh z - 1)I_0\left(\frac{\beta}{2}\right)\right.$$

$$\left.+\sinh^2 z\sum_{s=\pm1}\sum_{n=1}^{\infty} e^{-nz}\left[I_0\left(nz+\frac{s\beta}{2}\right)-\mathbf{L}_0\left(nz+\frac{s\beta}{2}\right)\right]\right\}. \tag{4.4}$$

In passing to the last line use has been made of the formula

$$\int_0^1 du\,\frac{e^{-zu}}{(1-u^2)^{1/2}} = \frac{\pi}{2}(I_0(z)-\mathbf{L}_0(z)) \tag{4.5}$$

where I_0 is the modified Bessel function of the first kind and $\mathbf{L}_0$ is the modified Struve function. Using the expression for $\cos t$ similar to (4.2) and utilising the formula

$$\int_0^{\infty} du\,\frac{e^{izu}}{(1+u^2)^{1/2}} = K_0(|z|)+\frac{\pi i}{2}(S(z)I_0(z)-\mathbf{L}_0(z)) \tag{4.6}$$

we can calculate the second term in the bracket of (2.5). Combining together, we reach

$$M_\perp^{(1)}(\omega_c) = \frac{\alpha\sqrt{\pi\beta}}{4\sinh(\beta/2)}(1-e^{-\beta\omega_c})\left\{-iI_0\left(\frac{\beta}{2}\right)+\frac{2}{\pi}\left[K_0\left(\frac{\beta}{2}\right)\right.\right.$$

$$\left.\left.+\cosh\frac{\beta\omega_c}{2}\sum_{s=\pm1}\sum_{n=1}^{\infty}\exp[-(n-1)\beta\omega_c/2]K_0[\tfrac{1}{2}\beta(n\omega_c+s)]\right]\right\}. \tag{4.7}$$

This means, if we write the absorption lineshape in a form (3.2), where now the mass in the front factor is replaced by the bare band mass m and the resonance is characterised by the cyclotron mass m_c to stress the high magnetic field case, then the cyclotron mass is given by

$$m_c = \{1+[\alpha\sqrt{\pi\beta}/4\omega_c\sinh(\beta/2)]I_0(\tfrac{1}{2}\beta)\}^{-1}. \tag{4.8}$$

The linewidth is

$$\Gamma = [\alpha/2\sinh(\beta/2)]\{(\beta/\pi)^{1/2}K_0(\beta/2)+[\cosh(\beta/2)]/\sqrt{\omega_c}\} \tag{4.9}$$

in the high magnetic field limit where we have used the asymptotic formula for $K_0(z)$. It should be noted that the cyclotron mass m_c at high field is smaller than the bare band mass and its asymptote at very low and high temperatures is given by

$$m_c^{-1} = \begin{cases} 1+(\alpha/2\omega_c)(1+1/4\beta+\ldots) & (\beta\gg1), \quad (4.10a)\\ 1+(\alpha/2\omega_c)(\pi/\beta)^{1/2}(1+\frac{1}{12}(\beta/2)^2+\ldots) & (\beta\ll1). \quad (4.10b)\end{cases}$$

This decrease of the cyclotron mass can be understood in terms of the so-called pinning effect (Larsen and Johnson 1966, Nakayama 1969). The low-temperature result indicates that the lowest-order perturbational correction to the separation between the Landau states $n=0$ and 1 is given by

$$\hbar\Delta \equiv E_1^{(1)} - E_0^{(1)} = (\alpha/2)\hbar\Omega \tag{4.11}$$

for high magnetic fields. In fact, it can be proved more generally that for any magnetic field, the perturbational result for the energy separation agrees precisely with $-\hbar\,\mathrm{Im}\,M_\perp^{(1)}(\omega_c)$ at zero temperature. Namely, our formula for m_c at zero temperature is exact to the first order in Γ_q^2 for all magnetic fields. The proof is given in the next section.

The linewidth consists of two contributions: the first term from the intra-band scattering, namely scattering within the same Landau sub-band index, and the other from the inter-band scattering. At low temperature, the inter-band scattering is more important. This is because the relaxation of the lowest Landau state is proportional to the phonon number which is exponentially small for optical phonons, while the decay channel of the $n=1$ state is always open irrespective of temperature, though this process is reduced by the density of states factor $\omega_c^{-1/2}$. It should be emphasised that Γ is not a simple function of the inverse lifetimes of the two lowest Landau states, because the relaxation time of the current is in general different from these lifetimes. This point was first emphasised by Kawabata (1967).

Next, we consider the strong coupling limit. When $\alpha \gg 1$ and $\omega_c \gg \alpha^2 \gg \beta^{-1}$, it is known (Saitoh 1981) that the approximate forms for $\Lambda_\perp(\tau)$ and $\Lambda_\parallel(\tau)$ are given by

$$\Lambda_\perp(\tau)=\frac{1}{\omega_c}\left(\frac{\cosh(\gamma\beta/2)-\cosh\gamma(\tau-\beta/2)}{\sinh(\gamma\beta/2)}+\frac{\cosh(\delta\beta/2)-\cosh\delta(\tau-\beta/2)}{\sinh(\delta\beta/2)}\right), \tag{4.12}$$

$$\Lambda_\parallel(\tau)=\frac{1}{v_H^2}\left[\tau\left(1-\frac{\tau}{\beta}\right)+\frac{v_H^2-1}{v_H}\,\frac{\cosh(\beta v_H/2)-\cosh v_H(\tau-\beta/2)}{\sinh(v_H\beta/2)}\right], \tag{4.13}$$

where

$$\gamma=\omega_c(1+\alpha^2\lambda/2\pi\omega_c), \tag{4.14}$$

$$\delta=\alpha^2\lambda/2\pi, \tag{4.15}$$

$$v_H=\alpha^2\lambda^2/\pi, \tag{4.16}$$

with

$$\lambda=\ln(2\pi\omega_c/e^2\alpha^2). \tag{4.17}$$

Since $\gamma \gg \delta \gg 1$, we simplify $\Lambda_\perp(\tau)$ as

$$\Lambda_\_(\tau)=(2/\omega_c)[1-\cosh\gamma(\tau-\beta/2)/2\cosh(\gamma\beta/2)] \tag{4.18}$$

and anticipating that the resonance occurs at $\omega \sim \gamma \gg \omega_c$, we disregard the term $\cosh(v_H(\tau-\beta/2))$ in $\Lambda_\parallel(\tau)$ and employ the form

$$\Lambda_\parallel(\tau)=[b^2-(\tau-\beta/2)^2]/\beta v_H^2 \tag{4.19}$$

where

$$b^2=\beta\{\tfrac{1}{4}\beta+[(v_H^2-1)/v_H]\coth\tfrac{1}{2}v_H\beta\}\approx\beta(v_H+\beta/4). \tag{4.20}$$

If we put (4.18) and (4.19) into (4.1) and then into (2.5), we have the expression $M_\perp(\omega)$ in a high magnetic field. For the first integral and imaginary part of the second term in (2.5), we may set

$$\Lambda_\parallel\approx b^2/\beta v_H^2 \tag{4.21}$$

since $b \gg 1$. Using the formula (4.2) and a similar expression for cos t, we reach after some algebra

$$M_{\perp}(\omega)=\frac{\alpha\omega_c}{2\omega}\left(\frac{\beta v_H^2}{\pi b^2}\right)^{1/2}\bigg[\mathrm{i}\omega^2\bigg(\frac{1}{1-\omega^2}+\frac{1}{2\sinh(\beta/2)}$$

$$\times\sum_{s=\pm 1}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{s\beta/2}}{2^n(n\gamma+s)[(n\gamma+s)^2-\omega^2]}+\mathrm{O}(\mathrm{e}^{-\beta\omega_c/2})\bigg)$$

$$+\frac{b\sinh(\beta\omega/2)}{2\sinh(\beta/2)}\sum_{s=\pm 1}\bigg(K_0[b(\omega+s)]+\frac{\mathrm{e}^{\beta\omega_c/2}}{2}K_0(b|\gamma-\omega+s|)$$

$$+\mathrm{O}(\mathrm{e}^{-\beta\omega_c-b\omega})\bigg)\bigg]. \tag{4.22}$$

In order to find the resonance point

$$\omega-\omega_c+\mathrm{Im}\,M_{\perp}(\omega)=0 \tag{4.23}$$

we retain the first term and $n=1$ terms in the sum of $\mathrm{Im}\,M_{\perp}(\omega)$ which might become large at $\omega\sim\gamma$, the remaining sum being of the order of γ^{-1} and negligible. The answer is given by

$$\omega=\gamma+\mathrm{O}(\alpha\lambda^{1/2}) \tag{4.24}$$

when $v_H\gg\beta/4$. Therefore $(\omega-\gamma)^{-1}=\mathrm{O}(\alpha^{-1}\lambda^{-1/2})\ll 1$ and the first term in the sum of $\mathrm{Im}\,M_{\perp}(\omega)$ is negligible in spite of its divergent appearance. This is equivalent to saying that the sum in $\mathrm{Im}\,M_{\perp}(\omega)$ can be discarded from the start. In other words, the intra-band scattering is unimportant in the strong coupling limit in contrast to the weak coupling case. The cyclotron mass to the accuracy α^2 is given by

$$m_c=(1+\alpha^2\lambda/2\pi\omega_c)^{-1} \tag{4.25}$$

which is exactly the same result that was inferred before from the free energy expression (Saitoh 1981). The cyclotron mass is again smaller than the bare band mass. This can be understood qualitatively in the following way. In the strong coupling case, the phonon polarisation field acts on an electron as an effective attractive potential so that the energy separation between the ground state and the excited states will in general be increased.

The linewidth is given by

$$\Gamma=\tfrac{1}{8}\alpha v_H(\beta/\pi)^{1/2}K_0(b)/\sinh(\beta/2). \tag{4.26}$$

The linewidth mainly comes from the intra-band scattering. The linewidth is very narrow in the strong coupling limit, indicating that the most important role of the electron–phonon coupling is to produce the self-trapping attractive potential and the residual scattering effect is small. A similar effect was observed in the low magnetic field case.

5. Discussion and summary

In order to understand the physical meaning of the formula (2.5) at a high magnetic field, let us consider the weak coupling limit. As will be shown in the appendix, our result can be written in a form

$$(m/2N_e e^2\hbar)\sigma_{+-}(\omega)=[\mathrm{i}(\omega-\omega_c-\langle\Delta(k)\rangle_e)+\langle\tau_c^{-1}(k)\rangle_e]^{-1} \tag{5.1}$$

where angular brackets indicate the thermal average over k as defined by (A10) and $\hbar\Delta(k)$ is the lowest-order perturbational correction due to the electron–phonon interaction to the Landau splitting between the states $n=0$ and 1,

$$\Delta(k)=(E_1^{(1)}(k)-E_0^{(1)}(k))/\hbar. \tag{5.2}$$

$\tau_c^{-1}(k)$ is the linewidth function given by

$$\tau_c^{-1}(k)=\frac{\pi}{\hbar}\sum_q\frac{\hbar q_\perp^2}{2m\omega_c}\Gamma_q^2\frac{\exp(-\hbar q_\perp^2/2m\omega_c)}{\sinh(\hbar\Omega_q\beta/2)}$$
$$\times\sum_{s=\pm1}\exp^{(s\hbar\Omega_q\beta/2)}\left(\frac{\hbar q_\perp^2}{2m\omega_c}\delta(E_0^{(0)}(k)-E_0^{(0)}(k+q_z)-s\hbar\Omega_q)\right.$$
$$\left.+\delta(E_1^{(0)}(k)-E_0^{(0)}(k+q_z)-s\hbar\Omega_q)\right) \tag{5.3}$$

with $E_n^{(0)}(k)$ the unperturbed Landau energy in the state n. The cyclotron mass is related to the separation between the lowest and the first Landau states which were modified by the electron–phonon interaction. The linewidth $\langle(\tau_c^{-1}(k)\rangle_e$ on the other hand is not a simple function of the inverse of the lifetimes of the two lowest states because of the modification brought by the current matrix element. In particular, when Γ_q^2 is long ranged, the contribution of the small-wavevector scattering to the current relaxation is reduced appreciably. This phenomenon is sometimes called the $(1-\cos\theta)$ reduction in analogy to the transport process in the no magnetic field case (Mott and Jones 1958).

Kawabata (1967) obtained a slightly different result from (5.2) in the case of weak electron–phonon interaction with elastic scattering. In order to appreciate the difference, let us introduce the mass operator $\hat{M}_\perp(\omega_c)$ symbolically written as

$$(m/2N_e e^2\hbar)\sigma_{+-}(\omega)=\langle\langle[\mathrm{i}(\omega-\omega_c)+\hat{M}_\perp(\omega_c)]^{-1}\rangle_{\mathrm{ph}}\rangle_e \tag{5.4}$$

where the outside and inside brackets indicate, respectively, the appropriate canonical average over electron and phonon variables. He calculated the mass operator $\hat{M}_\perp(\omega_c)$ to the lowest order in Γ_q^2 by Mori's method (1966) and then performed the average over phonon variables. Since at this stage the average could not be performed rigorously, he replaced (5.4) by the following approximate form

$$\langle[\mathrm{i}(\omega-\omega_c)+\langle\hat{M}_\perp(\omega_c)\rangle_{\mathrm{ph}}]^{-1}\rangle_e \tag{5.5}$$

and obtained the result

$$\langle\hat{M}(\omega_c)\rangle_{\mathrm{ph}}=-\mathrm{i}\Delta(k)+\tau_c^{-1}(k) \tag{5.6}$$

where $\Delta(k)$ and $\tau_c^{-1}(k)$ are the same functions defined above in (5.2) and (5.3). This formula is a bit different from ours (5.1) in the way of the thermal average over electron variables. Whether the approximation in passing from (5.4) to (5.5) is legitimate or not is an open question. In our derivation which results in (5.1) from (5.4) (Saitoh 1981), on the other hand, the average over phonon variables was performed rigorously first because of the very advantage of the path-integral method, and then a perturbation calculation was made for $\hat{M}_\perp(\omega_c)$. Since no rigorous justification for both the results is available to date to the best of the author's knowledge, judgement as to which form is right remains for future study.

If we disregard the above uncertainty, our formula (2.5) is most convenient to calculate the lineshape of the cyclotron resonance in the sense that there is no restriction on the strength of Γ_q^2 or inelasticity of scattering. We know no other theory which can treat the linewidth of the cyclotron resonance of a polaron strongly coupled to phonons with inelastic scattering on a unified way as presented here. It should be remarked also that at least at zero temperature our formula for the cyclotron resonance mass is exact to the lowest order in Γ_q^2 for any magnetic field. For, when the temperature is sufficiently low, from (5.1) we have

$$\operatorname{Im} M_{\perp}(\omega_c) = -\Delta(k=0). \tag{5.7}$$

On the other hand, the left-hand side is by definition equal to $(1-m/m_c)\omega_c$. Therefore, $\hbar\omega_c m/m_c$ becomes $\hbar\omega_c + \hbar\Delta(k=0)$ which is the energy separation between the lowest and first Landau states as it should be.

In conclusion, we have presented formulae for the absorption lineshape of the cyclotron resonance of a polaron. The present formula which is based on the path-integral method has the advantage over other theories that it is applicable to any type of electron–phonon interaction, whether they are strong or weak, elastic or inelastic. Existing theories are capable of treating only weak and elastic scatterings. Applications were made to the Fröhlich polaron system, and analytic expressions for the cyclotron mass shift and the lineshape were obtained for limiting cases of weak and strong couplings and low and high magnetic fields. The results for high magnetic fields are new.

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Appendix. General form of $M_{\perp}(\omega_c)$ in the weak coupling limit

Here, we shall obtain $M_{\perp}^{(1)}(\omega_c)$, the term proportional to Γ_q^2 in the perturbational expansion of $M_{\perp}(\omega_c)$. We start with the formula (2.5) where $L_{\perp}(\tau)$ is given by (2.6) with $\Lambda_{\perp}$ and $\Lambda_{\parallel}$ replaced by the zeroth approximants (2.8) and (2.9). We note the identity

$$\exp\left(\frac{\hbar^2 q_z^2}{2m\beta}\tau^2\right) = \frac{1}{Z}\sum_k \exp\left(-\frac{\beta\hbar^2 k^2}{2m} - \frac{\hbar^2 k q_z}{m}\tau\right) \tag{A1}$$

where the sum runs over the one-dimensional wavevector k and

$$Z = \sum_k \exp(-\beta\hbar^2 k^2/2m). \tag{A2}$$

Using this trick, we can linearise the τ^2 term in $\exp(-\hbar^2 q_z^2 \Lambda^{(0)}(\tau)/2m)$ which appears in the first integral in the bracket of (2.5). Applying a similar operation on $\exp[-\hbar^2 q_z^2 \Lambda^{(0)}(\mathrm{i}t+\beta/2)/2m]$, we have

$$M_\perp^{(1)}(\omega_c) = \frac{1}{\hbar Z}\sum_k \exp(-\beta\hbar^2 k^2/2m)\sum_q \frac{x^2\Gamma_q^2}{\sinh(\hbar\Omega_q\beta/2)}\exp\left(-x^2\coth z - \frac{\beta\varepsilon}{2}\right)$$
$$\times \sum_{s=\pm1}\left(\mathrm{i}\int_0^{\beta/2} \mathrm{d}\tau\,[\cosh\hbar\omega_c(\tau-\beta/2)-1]\exp[y\cosh\hbar\omega_c\tau - (\varepsilon+s\hbar\Omega_q)\tau]\right.$$
$$\left. + \sinh z\int_0^\infty \mathrm{d}t\,\exp(-\mathrm{i}\hbar\omega_c t)\exp[y\cos\hbar\omega_c t - \mathrm{i}(\varepsilon+s\hbar\Omega_q)t]\right) \tag{A3}$$

where z is defined as before by (4.3) in the text, and the following abbreviations have been used:

$$\varepsilon = \hbar^2(q_z^2+2kq_z)/2m \tag{A4}$$
$$x^2 = \hbar q_\perp^2/2m\omega_c, \tag{A5}$$
$$y = \hbar q_\perp^2/[2m\omega_c\sinh(\hbar\omega_c\beta/2)]. \tag{A6}$$

Noting the identity

$$\exp(y\cosh\tau) = I_0(y) + 2\sum_{n=1}^\infty I_n(y)\cosh n\tau \tag{A7}$$

where I_n is the modified Bessel function of the first kind, we can expand $\exp(y\cosh\hbar\omega_c\tau)$ in a series in $\cosh n\hbar\omega_c\tau$ and perform the τ integration term by term. Using a similar expression for $\exp(y\cos t)$, we reach the result

$$\mathrm{Re}\,M_\perp^{(1)}(\omega_c) = \left\langle \frac{\pi}{\hbar}\sum_q x^2\Gamma_q^2\exp(-x^2\coth z)\frac{\sinh z}{\sinh(\hbar\Omega_q\beta/2)}\sum_{s=\pm1}\exp(s\hbar\Omega_q\beta/2)\right.$$
$$\left.\times\left(I_1(y)\delta(\varepsilon_s) + \sum_{n=1}^\infty (I_{n-1}(y)+I_{n+1}(y))\,\mathrm{e}^{-nz}\delta(\varepsilon_s - n\hbar\omega_c)\right)\right\rangle_{\mathrm{e}}, \tag{A8a}$$

$$\mathrm{Im}\,M_\perp^{(1)}(\omega_c) = \left\langle \frac{1}{\hbar}\sum_q x^2\Gamma_q^2\exp(-x^2\coth z)\frac{1}{\sinh(\hbar\Omega_q\beta/2)}\right.$$
$$\times\sum_{s=\pm1}\exp(-s\hbar\Omega_q\beta/2)\left[\frac{1}{\varepsilon_s}(\cosh z I_1(y) - I_0(y))\right.$$
$$\left.\left. + \sum_{r=\pm1}\sum_{n=1}^\infty \frac{\mathrm{e}^{rns}}{\varepsilon_s + rn\hbar\omega_c}\left(\frac{\mathrm{e}^{-rz}I_{n-1}(y)+\mathrm{e}^{rz}I_{n+1}(y)}{2} - I_n(y)\right)\right]\right\rangle_{\mathrm{e}}, \tag{A8b}$$

where $s = +1$ or -1 corresponds, respectively, to a phonon emission or an absorption process, and

$$\varepsilon_s = \varepsilon + s\hbar\Omega_q \tag{A9}$$

and the brackets denote the thermal average for the one-dimensional degree of freedom

$$\langle\cdot\rangle_{\mathrm{e}} = Z^{-1}\sum_k(\cdot)\exp(-\beta\hbar^2k^2/2m). \tag{A10}$$

In particular, when $z=\hbar\omega_c\beta/2\gg 1$, namely the case of either very low temperature or very high magnetic fields, we may replace $I_n(z)$ by its first term of the expansion

$$I_n(y)=x^{2n}\,\mathrm{e}^{-nz}/n! \tag{A11}$$

and therefore

$$\operatorname{Re} M_\perp^{(1)}(\omega_c)=\Big\langle\frac{\pi}{\hbar}\sum_q\frac{x^2\Gamma_q^2\exp(-x^2)}{2\sinh(\hbar\Omega_q\beta/2)}$$

$$\times\sum_{s=\pm1}\exp(s\hbar\Omega_q\beta/2)[x^2\delta(\varepsilon_s)+\delta(\varepsilon_s-\hbar\omega_c)]\Big\rangle_{\mathrm{e}}, \tag{A12a}$$

$$\operatorname{Im} M_\perp^{(1)}(\omega_c)=\Big\langle\frac{1}{\hbar}\sum_q\frac{x^2\Gamma_q^2\exp(-x^2)}{2\sinh(\hbar\Omega_q\beta/2)}\sum_{s=\pm1}\exp(s\hbar\Omega_q\beta/2)$$

$$\times\Big(\frac{1}{\varepsilon_s-\hbar\omega_c}+\frac{x^2-2}{\varepsilon_s}+\sum_{n=1}^{\infty}\frac{x^{2n-2}[x^4-2(n+1)x^2+n(n+1)]}{(n+1)!(\varepsilon_s+n\hbar\omega_c)}\Big)\Big\rangle_{\mathrm{e}}. \tag{A12b}$$

In order to understand these results, let us consider the lowest perturbational correction to the lowest two Landau energies. They are given respectively by (Larsen 1972)

$$E_0^{(1)}(k)=-\sum_q\frac{\Gamma_q^2\exp(-x^2)}{2\sinh(\hbar\Omega_q\beta/2)}\sum_{s=\pm1}\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}\frac{\exp(s\hbar\Omega_q\beta/2)}{\varepsilon_s+n\hbar\omega_c} \tag{A13}$$

$$E_1^{(1)}(k)=-\sum_q\frac{\Gamma_q^2\exp(-x^2)}{2\sinh(\hbar\Omega_q\beta/2)}\sum_{s=\pm1}\sum_{n=0}^{\infty}\frac{x^{2n-2}}{n!}\frac{(x^2-n)^2\exp(s\hbar\Omega_q\beta/2)}{\varepsilon_s+(n-1)\hbar\omega_c}. \tag{A14}$$

Therefore, comparing (A13) and (A14) with (A12*b*), we reach

$$\operatorname{Im} M_\perp^{(1)}(\omega_c)=\langle E_0^{(1)}(k)-E_1^{(1)}(k)\rangle_{\mathrm{e}}/\hbar. \tag{A15}$$

The linewidth cannot be interpreted simply as the inverse of the lifetimes of the $n=0$ and 1 states. This is due to the modifications brought by the current matrix elements which are different in principle from the matrix elements for the lifetime.

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